Advanced and Contemporary Topics in Macroeconomics I

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Class Lecture Note 7

Investment

Based on Eklund(2013), Hib's Lectures & David Romer (2012) Book

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Lecture 7
Optimal Investment
1. Introduction: Why Care about Investment

1. Why Care About Investment?

- Investment drives capital formation, and the stock of capital is a key determinant of output growth and consequently feasible consumption levels.

\[ K_{t+1} - K_t = I_t - \delta K_t \]  
\[ K_{t+1} = (1 - \delta)K_t + I_t \]
• Investment is 15% or more of GDP and it is the most volatile component of GDP.
• Investment therefore is probably the most important proximate source of business cycles [This is for instance central for Kalecki/Keynes].
• Government policy is typically targeted heavily on investment; most tax codes favor it.
• Investment dynamics yield information about investor rationality, Keynes. "animal spirits", capital formation institutions, and more.
• Thus, the need to focus on it.
2. Static Neoclassical Investment Theory

• Neoclassical investment theory dates mainly from Dale Jorgensen’s papers in the 1960s (AER, 1963, Hall and Jorgenson, AER, 1967)

• However, what Jorgenson delivered was more a theory of the optimal capital stock than a theory of optimal investment per se.

• Here is the essential setup: (see the next slides)
• Recall [from ur graduate/undrgad macro]

• Output is produced by a CRS production technology

\[ Q = F(K, A \bullet N) \]

• Firms invest to achieve a capital stock that maximizes net real profit, \( \pi \), each period

\[ \pi_t = F(K_t, A \bullet N_t) - r_K K_t - w \bar{N} \]

• where everything is priced in units of output**

\[ r_K = p_K \cdot \left( r + \delta - \frac{\Delta p_K}{p_K} \right) \]

is the user cost of capital given the real price of capital units \( p_K \); a fixed market interest rate \( r \); and depreciation rate \( \delta \), and \( w \) is the real wage.

– \( r_K/p_K \) gives us the real user cost of capital.
1. BROADER DERIVATION OF NEOCLASSICAL THEORY OF INVESTMENT

• In this section we derive the relationship between the neoclassical theory, accelerator principle and Tobin’s Q-theory of investment.

• All three theories assume optimization behavior on behalf of the decision maker (investor).

• The neoclassical and Tobin’s theory of investment explicitly assumes profit/value maximization.

• The accelerator theory of investment assumes this implicitly, by assuming that investment is determined by an optimal capital stock.
The starting point for Jorgenson’s (1963, 1967 and 1971) neoclassical investment theory is the optimization problem of a firm.

Maximizing profits in each period will yield an optimal capital stock.

Assuming that the production function can be written as a conventional Cobb-Douglas function:

\[ Y(t) = f(K(t), L(t)) = AK^\alpha L^{1-\alpha} \]  

Where \( Y(t) \) is firm output, \( K \) is capital and \( L \) denotes labor, all in period \( t \).
The profit function for a representative firm can then be expressed as follows:

\[ \pi(t) = p(t)Y(t) - s(t)I(t) - w(t)L(t) \]  

\( \pi(t) \) denotes profit, \( p(t) \) is the price of output, \( s(t) \) is the price of capital and \( w(t) \) is the wage.

Assuming profit maximization, the current value of a firm, \( V(0) \), can be written as:

\[
V(0) = \max E_{\Phi_0} \int \pi(t) e^{-rt} dt \\
= E_{\Phi_0} \int_0^\infty \left[ p(t)Y(t) - s(t)I(t) - w(t)L(t) \right] e^{-rt} dt
\]  

\( S.t \quad \frac{dK}{dt} = I(t) - \delta K(t) = \dot{K}(t) \) and \( K(0) \) is given.
To maximize $V(0)$ the first step is to set up a Lagrangian:

$$L = V(0) + \int_0^\infty \lambda [I - \delta K - \dot{K}] e^{-rt} dt$$ \hspace{1cm} (5)

Which gives

$$L = \int_0^\infty \left[ pY - sI - wL + \lambda \dot{K} \right] e^{-rt} dt$$ \hspace{1cm} (6)

From this we obtain the familiar current value Hamiltonian

$$H = pf(K, L) - sI - wL + \lambda (I - \delta K)$$ \hspace{1cm} (7)

Where the Lagrangian multiplier $\lambda(t)$ is our costate variable. It should be noted that $\lambda(t)$ represents the shadow price of capital.
Differentiating the Hamiltonian, we obtain the following first order conditions:

\[
\frac{\partial H}{\partial I} = -s + \lambda = 0
\]  

(8)

This condition states that the opportunity cost of capital shall be equal to the shadow price of capital.

\[
\frac{\partial H}{\partial L} = pf'_L - w = 0
\]  

(9)

This condition simply says that the labor should be employed until the marginal revenue of labor equates with the wage.
Recalling the maximum principle (Intriligator, 1971) we get:**
\[
\frac{\partial H}{\partial \lambda} = \frac{\partial K}{\partial t} = I - \delta K
\]  \hspace{1cm} (10)

(this is corrected now thanks to Marti’s comment in class)

Which says that in equilibrium net investment should be zero and gross investment equal to the depreciation of \( K \).

Finally, the marginal condition for capital is:
\[
\frac{\partial H}{\partial K} = pf_k' - \lambda \delta = 0
\]  \hspace{1cm} (11)

The canonical equation (Intriligator, 1971, Chinag 2000) requires that
\[
\dot{q} = -\frac{\partial H}{\partial K}, \text{ where } y \text{ is the control variable such that } q = \lambda e^{-rt} \text{ at time } t. \text{ [**Recall the 2nd FOC in Lecture 2]}
• Thus:
\[
- \frac{\partial H}{\partial K} = \frac{d}{dt}[e^{-rt}\lambda(t)] = \frac{\partial \lambda}{\partial t} - r\lambda
\]  
(12)

• This means that equation (11) can be written as:
\[
- pf_K' + \lambda \delta = \frac{\partial \lambda}{\partial t} - r\lambda
\]  
(13)

• From equation (8) we know that \( s=\lambda \), which implies that \( \frac{\partial s}{\partial t} = \frac{\partial \lambda}{\partial t} \)

• This also means that \( \frac{\partial H}{\partial K} \) can be stated in the following way:
\[
- pf_K' + s\delta = \frac{\partial s}{\partial t} - rs
\]  
(14)
Rearranging the above we obtain:

\[ pf'_K = s \left[ \delta + r - \left( \frac{\partial s}{\partial t} \right) / s \right] \] (15)

Since \( pf'_K \) is the marginal rate of return on capital, \( mrr_{K^\prime} \), equation (15) can be rewritten as the marginal product of capital:

\[ f'_K = s \left[ \delta + r - \left( \frac{\partial s}{\partial t} \right) / s \right] / p \] (16)
Note that \( f'_K = \frac{\partial Y}{\partial K} \). Thus Jorgenson’s (1963) user cost of capital, \( c \), is defined as: 
\[ s[\delta + r - (\partial s/\partial t) / s] \]
which means that:
\[ p f'_K = c \quad \text{or} \quad f'_K = c / p \text{ in real terms} \quad (17) \]

This can now be used to derive the optimal capital stock, \( K^* \), and the investment function. Using Cobb-Douglas technology the marginal product of capital becomes:

\[ \frac{\partial Y}{\partial K} = f'_K = \alpha K^{\alpha-1} L^{1-\alpha} \quad (18) \]

Which in turn can be expressed as:

\[ \frac{\partial Y}{\partial K} = \alpha \frac{Y}{K} \quad (19) \]
Multiplying by $p_t$ and recalling equation (17) we get:

\[
\frac{\partial H}{\partial K} = p \frac{\alpha Y}{K} = c \tag{20}
\]

Solving for $K$ we obtain an expression for the optimal capital stock:

\[
K^* = \frac{p\alpha Y}{c} \tag{21}
\]

We see that $K^*$ depends on output, price of output and the user cost of capital, $c$.

Thus, investment becomes the change in capital between two periods:

\[
I = \frac{p\alpha Y}{c} - K^* (t - \tau) \tag{22}
\]
Note that

– this assumes that $K(t)$ adjusts instantaneously and fully to $K^*(t)$.
– Assuming that the adjustment to the optimal capital stock is only partial each period this can be incorporated into equation (22) by introducing an adjustment parameter that depends on the difference between actual and desired capital, see e.g. Mueller (2003).

– Since the neoclassical theory assumes that the capital adjusts immediately and completely to the desired capital stock the investment function is essentially eliminated.
  • It has therefore been suggested that Jorgenson’s theory is in fact a capital theory and not an investment theory.
2.2 ACCELERATOR THEORY

• The accelerator approach is often associated with a Keynesian approach which is primarily due to its assumption of fixed prices.
• The acceleration principle was however first suggested by Clark (1917) and is well known for its applications by Samuelsson (1939) to business cycles.
• The accelerator is, in fact, merely a special case of the neoclassical theory of investment where the price variables have been reduced to constants.
• Recall the Equation [21] which is the optimal level of capital stock:

\[ K^* = \frac{p\alpha Y}{c} \]  (21)
If the price of output is assumed to be constant and the price variables $s$ and $r$ in Jorgenson’s (1963) user cost of capital, $(c = s[\delta + r - (\partial s/\partial t)/s])$, are fixed, equation (21) reduces to following:

$$K^* = \alpha Y$$

(23)

This is simply the well known accelerator principle where the desired capital stock is assumed to be proportional to output.

Investment in any period will therefore depend on the growth in output:

$$I = \alpha \dot{Y}$$

or in discrete set up

$$I = \alpha \Delta Y$$

(24)

Given flexible prices and partial adjustment toward the desired capital stock each period, investment depend on prices of output and input and interest rates (cost of capital).
Smith (1961) demonstrates what he calls the: “logical inseparability” of ‘marginal efficiency’ and the ‘accelerator’ determinant of investment expenditures”. He used calculus of variation to derive his results.

Again, this version of the accelerator assumes a complete and instantaneous adjustment of the capital stock.

An alternative is the so-called flexible accelerator that includes lags in the capital stock.

– Eisner and Strotz (1963) suggest that these lags are because the unit price of capital, $s(t)$, increases with the adjustment speed, (see also Lucas, 1967).

– Allowing for lags in the adjustment of the capital stock, however, make the neoclassical theory virtually indistinguishable from the accelerator theory (see Eisner and Strotz, 1963, and Eisner and Nadiri, 1968).

Furthermore, it is worthwhile to note that even though the accelerator principle is often coupled with a Keynesian approach, Keynes himself, was very skeptical towards approaches like this.

– First, Keynes was very critical towards formal models of economic behavior.

– Second, and more fundamentally, Keynes did not believe that investment is determined as adjustment towards equilibrium.
Remarks:

• The static neoclassical setup implies that investment needed to get K to KD occurs all at once - in one discrete jump. In general this would seem highly unrealistic.

• Jorgenson has actual output on the right-side of the investment equation. This is wrong. What should appear is desired output, QD.

• 1960s neoclassical theory is pre-rational expectations and, therefore, expectations of future profitability of increments to the capital stock today play no role in explaining today’s investment behavior.
3. Optimal Investment with Convex Adjustment Costs

*The State of the World:*

- The firm is a price taker in competitive markets.
- Labor is flexible (can be adjusted without cost).
- The only barrier to full and fast deployment of the profit maximizing stock of capital are the convex costs of adjusting the capital stock.
- Firms can always obtain the capital desired (the market for investment capital is perfect).
- There are no taxes.
- The production technology has CRS.
- As we shall see, introducing adjustment costs yields rich dynamics to investment behavior.
3.1 The Producer’s Objective

• The representative firm aims to maximize its financial value, as the expected present discounted value of its net real revenue flow - revenue from sales less capital investment costs and labor (and other) costs

\[
\text{Max}_{\{I_{t+j}, K_{t+j+1}\}_{j=0}^\infty} E_t \sum_{j=0}^\infty \beta^j \pi_{t+j} \\
\pi_t = F[K_t, (A \bullet N_t)] - I_t - C(I_t, K_t) - w_t N_t
\]

(18)

(19)

• Subject to

\[
K_{t+1} - K_t = I_t - \delta K_t
\]

(20)
implying that investment satisfies

\[ I_t = K_{t+1} - (1 - \delta)K_t \]  

(21)

Where:

• \( I_t \) is the real cost of capital goods
• \( C(I_t, K_t) \) represents real adjustment costs in the form of diminished revenue from output sales,
• \( \beta = \frac{1}{(1+r)}, r > 1 \) is the discount factor.
In order to get transparently to the famous "Tobin’s q", we set up the producer’s problem as a current value Lagrangian function

\[
L = \min_{\{I_{t+j}, K_{t+j+1}\}_{i=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j \left\{ F[K_{t+j}, (AN_{t+j})] - I_t - C(I_{t+j}, K_{t+j}) \right. \\
\left. - w_{t+j} N_{t+j} + q_{t+j} [I_{t+j} - K_{t+j+1} + (1 - \delta)K_{t+j}] \right\} 
\]

where \(E_t\) denotes conditional expectations informed by all outcomes at period \(t\) and earlier, and \(q_t\) (\(q_{t+j}\)) gives the value (shadow price) of a marginal change in \(K\) at time \(t + 1\) (\(t + j + 1\)) in time \(t\) (\(t + j\)) revenue.
3.2 The FOCs

- Because labor can be adjusted without cost, we need two FOCs \( I_t \) and \( K_{t+1} \).

(1) The Investment FOC

\[
\frac{\partial L}{\partial I_t} = -1 - C'(I_t) + q_t = 0 \quad (23)
\]

- Hence we obtain the shadow price of a new unit of capital as 1 (the normalized price of a unit of capital) plus the marginal cost of its installation:

\[
q_t = 1 + C'(I_t) \quad (24)
\]

Marginal Benefit\( (q_t) = \) Marginal Cost\( (1 + C'(I_t)) \)
To be more explicit we need to specify the cost function, $C(I,K)$. A common specification is the strictly convex quadratic function

$$C(I_t, K_t) = \frac{b}{2} \left( \frac{I_t}{K_t} - \bar{C} \right)^2 K_t$$

(25)

where $\bar{C}$ is the ‘costless’ level of installation (which could of course be zero) and $b$ is just a scale parameter.

[[ See next slide for Diagram]]
Optimal Investment......Cont’d
• In this fairly flexible case the shadow price is**

\[ q_t = 1 + C'(I_t) \]  \hspace{1cm} (26)

\[ = 1 + b \left( \frac{I_t}{K_t} - \bar{C} \right) \]

• As initially proposed by Tobin (JMCB, 1969), \( q \) is therefore a sufficient statistic for investment. The FOC for \( I_t \) in (26) implies the investment equation (investment relative to the existing stock of capital)

\[ \frac{I_t}{K_t} = \frac{1}{b} (q_t - 1) + \bar{C} \]  \hspace{1cm} (27)
Equation 27 is a famous equation, which is sometimes written with $b=1$ and $\bar{C}=0$.

It means that if the firm is maximizing it should invest up to the point at which the marginal cost of a new unit of capital equals its value (shadow price) to the PDV of its net real revenues.

When the value of a new unit of capital exceeds its price ($1$), investment should rise above the costless level $\bar{C}$, and (of course) conversely.
Optimal Investment......Cont’d

Capital Stock Dynamics

• Since

\[ \Delta K_{t+1} = I_t + \delta K_t \]

\[ \frac{\Delta K_{t+1}}{K_t} = \frac{1}{b} (q_t - 1) + (\bar{C} - \delta) \]  \hspace{1cm} (28)

(2) The Capital FOC

• The FOC is found for \( K_{t+1} \) because that is what is driven by \( I_t \). \((K_t \text{ is sunk.})^{**} \)

\[ \frac{\partial L}{\partial K_{t+1}} = E_t \left\{ \beta \left[ F'(K_{t+1}) - C'(K_{t+1}) \right] - q_t + \beta q_{t+1} (1 - \delta) \right\} = 0 \]  \hspace{1cm} (29)

\[ \Rightarrow q_t = E_t \beta \left[ F'(K_{t+1}) - C'(K_{t+1}) + q_{t+1} (1 - \delta) \right] \]  \hspace{1cm} (30)
The above equation is an investment Euler equation.

Note well its interpretation. It says that the marginal value (shadow price) of a new unit of capital stock today \( q_t \) must equal the expected discounted value (\( \beta = 1/(1+r) \)) of the sum of three effects:

- (i) the new unit’s marginal contribution to real revenue, \( F'(K_{t+1}) \),
- (ii) the contribution of the new unit to the marginal decline of future capital installation costs, \( -C'K_{t+1} \)
- (iii) the depreciated shadow value of capital in the next period, \( q_{t+1} (1-\delta) \)

If these conditions do not hold, the firm is missing a profitable intertemporal arbitrage opportunity.
• Now let’s re-arrange terms in (30) to get \( q_{t+1} \) on the left-side:

\[
q_t - E_t \beta q_{t+1} (1 - \delta) = E_t \beta [F'(K_{t+1}) - C'(K_{t+1})] \tag{31}
\]

\[
E_t q_t [1 - \beta (1 - \delta)L^{-1}] = E_t \beta [F'(K_{t+1}) - C'(K_{t+1})] \tag{32}
\]

• Now solve for \( q_t \):

\[
q_t = \frac{E_t \beta [F'(K_{t+1}) - C'(K_{t+1})]}{[1 - \beta (1 - \delta)L^{-1}]} \tag{33}
\]
• Enumerating the forward polynomial, we obtain:

\[
\frac{1}{[1 - \beta (1 - \delta)L^{-1}]} = \frac{1}{[1 - \frac{(1 - \delta)}{(1 + r)} L^{-1}]} 
\]

we obtain:

\[
q_t = E_t \frac{1}{(1 + r)} \sum_{j=0}^{\infty} \left( \frac{(1 - \delta)}{(1 + r)} \right)^j [F'(K_{t+1+j}) - C'(K_{t+1+j})] 
\]

Again we have a beautiful equation, and you should understand well its interpretation.

• The equation says that the marginal value (shadow price) of an extra unit of capital (q) must equal the extra unit’s contribution to the expected PDV of real net revenues, where the operative discount factor incorporates the rate at which capital depreciates each period:

\[
\frac{(1 - \delta)}{(1 + r)} < \frac{1}{(1 + r)} = \beta
\]
4. Tobin’s q and Stock Prices

• Hayashi (1982, Econometrica) proved that if the production technology and the adjustment cost function are homogeneous of degree 1 (CRS), as we have assumed here, then Tobin’s marginal q equals average q, and average q gives the market value of the firm relative to the replacement cost of its capital stock.

• To see this, engage in the following exercise. Let us go through all the algebraic steps (most of them redundant for you) to obtain the result.
• First, multiply the solution in eq.(30) for \( q_t \) by \( K_{t+1} \). This gives

\[
K_{t+1}q_t = E_t\beta \left[ \frac{F'(K_{t+1})K_{t+1}}{-C'(K_{t+1})K_{t+1} + q_{t+1}(1-\delta)K_{t+1}} \right]
\]

(35)

• By CRS we know that factor payments exhaust output

\[
Q = F(K, AN) = F'(K)K + F'(N)N,
\]
• so the first right-side term in brackets in eq.(35) is

\[ F'(K_{t+1})K_{t+1} = F(K_{t+1}, AN_{t+1}) - F'(N_{t+1})N_{t+1} \]

(36)

\[ = F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} \]

• Hence eq.(35) may be expressed

\[ K_{t+1}q_t = E_t \beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} \right. \]

\[ \left. - C'(K_{t+1})K_{t+1} + q_{t+1}(1-\delta)K_{t+1} \right] \]

(37)
• From the capital accumulation equation evaluated at period $t+2$ we know that

$$K_{t+2} - K_{t+1} = I_{t+1} - \delta K_{t+1}$$

• And so

$$(1 - \delta)K_{t+1} = K_{t+2} - I_{t+1}$$

• Therefore by definition

$$q_{t+1}(1 - \delta)K_{t+1} = q_{t+1}(K_{t+2} - I_{t+1})$$
• Now substituting $q_{t+1}(K_{t+2} - I_{t+1})$ for $q_{t+1}(1-\delta)K_{t+1}$ in eq.(37):

$$K_{t+1}q_t = E_t\beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} - C'(K_{t+1})K_{t+1} + q_{t+1}(K_{t+2} - I_{t+1}) \right]$$

$$= E_t\beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} - C'(K_{t+1})K_{t+1} + q_{t+1}K_{t+2} - q_{t+1}I_{t+1} \right]$$

(38)
• From the investment FOC [Eqn 24] we know that

\[ q_{t+1} = (1 + C')I_{t+1} \]

Make this substitution for the last term in eq.(38):

\[ K_{t+1}q_t = E_t \beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} - C'(K_{t+1})K_{t+1} + q_{t+1}K_{t+2} - [1 + C'(I_{t+1})]I_{t+1} \right] \]

\[ = E_t \beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} - C'(K_{t+1})K_{t+1} + q_{t+1}K_{t+2} - C'(I_{t+1})I_{t+1} - I_{t+1} \right] \]
Next remember our assumption (necessary for Hayashi’s proof) that the adjustment cost function, like, the production function, is homogenous of degree 1,

\[ C'(K_{t+1})K_{t+1} + C'(I_{t+1})I_{t+1} = C(I_{t+1}; K_{t+1}) \]

Make this substitution in eq.(39), giving

\[ K_{t+1}q_t = E_t \beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1} N_{t+1} \right. \\
\left. - C'(I_{t+1}, K_{t+1}) + q_{t+1} K_{t+2} - I_{t+1} \right] \] (40)
• We have obtained a difference equation for $K_{t+1}q_t$ with driving (forcing) function

$$
E_t \beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1}N_{t+1} 
- C(I_{t+1}, K_{t+1}) - I_{t+1} \right]
$$

• Note that the driving function is the net real revenue of the firm [eq.(19)]: output less wage costs less the costs of installation of capital less direct capital investment costs.
• Use the usual methods to solve (40) by first moving $E_t[\beta q_{t+1} K_{t+2}]$ to the left hand side

$$E_t K_{t+1} q_t (1 - \beta L^{-1}) = E_t \beta \left[ F(K_{t+1}, AN_{t+1}) - w_{t+1} N_{t+1} \right]$$

$$- C(I_{t+1}, K_{t+1}) - I_{t+1}$$

(41)

• and then divide through by and enumerate the forward polynomial operator $(1 - \beta L^{-1})$:

$$K_{t+1} q_t = E_t \beta \sum_{j=0}^{\infty} \beta^j \left[ F(K_{t+1+j}, AN_{t+1+j}) - w_{t+1+j} N_{t+1+j} \right]$$

$$- C(I_{t+1+j}, K_{t+1+j}) - I_{t+1+j}$$

(42)
• Next divide through by $K_{t+1}$

$$q_t = E_t \left\{ \beta \sum_{j=0}^{\infty} \beta^j \left[ F(K_{t+1+j}, AN_{t+1+j}) - w_{t+1+j} N_{t+1+j} \right] \right. \left. - C(I_{t+1+j}, K_{t+1+j}) - I_{t+1+j} \right\} \frac{1}{K_{t+1}} \right\} \right\}$$

(43)

• Finally we have arrived at yet another important equation of the Tobin-q model!

• It says that $q_t$ is the ratio of the expected present discounted value of the firm’s revenue flows relative to the market value of its capital stock. (Recall the price of capital was normalized to 1.0).
The numerator gives the ‘fundamental’ market value of a firm:
- the expected present value of its revenue flow (equal to its price per share times the number shares outstanding), if asset markets are efficient.

The denominator is the replacement cost of its capital stock.

If producer’s are investing optimally, the shadow price of an extra unit of investment (an extra unit of capital) should be equated to the firm’s market value per unit of capital value

[ie. I is =f(q)] – the Tobin-q investment model

Read on “Investment and the growth of capital stock –using phase diagram” in your Textbook.
The End
Thank You